Apply the following standard approaches for basic proofs about sets.

- To show an if and only if statement, we show each direction of implication separately; that is, to show that $p \Rightarrow q$, show that $p \Rightarrow q$ and $q \Rightarrow p$.
- To show an implication, we assume the hypothesis, and show how that implies the conclusion; that is, to show that $p \Rightarrow q$, assume p, and (using definitions and propositions) show q.
- To show that two sets are equal, we show that each is contained in the other; that is, to show that $A = B$, show that $A \subset B$ and $B \subset A$.
- To show that A is contained in B, we select an arbitrary element in A, and (using definitions and propositions) show that it is in B.

Problem 1. Let A and B be a sets. Show that

$$A \subset B \quad \Leftrightarrow \quad A \cap B = A.$$

Proof. ($\Rightarrow$) Here we show that if $A \subset B$, then $A \cap B = A$.

Suppose that $A \subset B$. We wish to show that $A \cap B = A$. To do this, we show that $A \cap B \subset A$ and $A \subset A \cap B$.

($\subset$) Here we show that $A \cap B \subset A$.

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus $x \in A$. Therefore $A \cap B \subset A$.

($\supset$) Here we show that $A \cap B \supset A$, that is, that $A \subset A \cap B$.

Let $x \in A$. Since $A \subset B$, $x \in B$. Thus $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $A \subset A \cap B$.

We have shown that $A \subset B$ implies $A \cap B = A$.

($\Leftarrow$) Here we show that if $A \cap B = A$, then $A \subset B$.

Suppose that $A \cap B = A$. We wish to show that $A \subset B$.

Let $x \in A$. Then, since $A = A \cap B$, we know that $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular, $x \in B$. Thus $A \subset B$.

We have shown that $A \cap B = A$ implies $A \subset B$. □

Here is the same proof, boiled down a little.

Proof. We show both directions of the implication.

($\Rightarrow$) Suppose that $A \subset B$. We wish to show that $A \cap B = A$.

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus $x \in A$. Therefore $A \cap B \subset A$.

Let $x \in A$. Since $A \subset B$, $x \in B$. Thus $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $A \subset A \cap B$. It follows that $A \cap B = A$.

($\Leftarrow$) Suppose that $A \cap B = A$. We wish to show that $A \subset B$.

Let $x \in A$. Then, since $A = A \cap B$, we know that $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular, $x \in B$. Thus $A \subset B$. □

Problem 2. Let A and B be a sets. Show that

$$A \subset B \quad \Leftrightarrow \quad A \cup B = B.$$

Solution. We show both directions of the implication.

($\Rightarrow$) Suppose that $A \subset B$. We wish to show that $A \cup B = B$.

Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$, since $A \subset B$. On the other hand, if $x \in B$, then x is already in B. So in either case, $x \in A$. Therefore $A \cup B \subset B$.

Let $x \in B$. Then $x \in A$ or $x \in B$, so $x \in A \cup B$. Thus $B \subset A \cup B$. It follows that $A \cup B = B$.

($\Leftarrow$) Suppose that $A \cup B = B$. We wish to show that $A \subset B$.

Let $x \in A$. Then, $x \in A$ or $x \in B$, so $x \in A \cup B$. Since $A \cup B = B$, $x \in B$. Thus $A \subset B$. □

For each of the problems below, draw a Venn diagram.

Problem 3. Let A, B, C be sets. Show that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

Proof. We show both directions of inclusion.

($\subset$) Suppose that $x \in (A \cup B) \cap C$. [WTS: $x \in (A \cap C) \cup (B \cap C)$]

Then $x \in A \cup B$ and $x \in C$. Thus $x \in A$ or $x \in B$, and $x \in C$.

If $x \in A$, then $x \in A$ and $x \in C$, so $x \in A \cap C$.

Otherwise, if $x \in B$, then $x \in B$ and $x \in C$, so $x \in B \cap C$.

Thus either $x \in A \cap C$ or $x \in B \cap C$, whence $x \in (A \cap C) \cup (B \cap C)$.

($\supset$) Suppose that $x \in (A \cap C) \cup (B \cap C)$. [WTS: $x \in (A \cup B) \cap C$]

Then $x \in A \cap C$ or $x \in B \cap C$.

If $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in A$, $x \in A \cup B$. Thus $x \in A \cup B$ and $x \in C$, so $x \in (A \cup B) \cap C$.

Otherwise, $x \in B \cap C$, so $x \in B$ and $x \in C$. Since $x \in B$, $x \in A \cup B$. Thus $x \in A \cup B$ and $x \in C$, so $x \in (A \cup B) \cap C$.

In either case, $x \in (A \cup B) \cap C$. □

Problem 4. Let A, B, C be sets. Show that

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Solution. We show containment in both directions.

($\subset$) Let $x \in (A \cap B) \cup C$. Then $x \in A \cap B$, or $x \in C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, whence $x \in A \cup C$ and $x \in B \cup C$, which says that $x \in (A \cup B) \cap (B \cup C)$. On the other hand, if $x \in C$, then $x \in A \cup C$ and $x \in B \cup C$, so $x \in (A \cup B) \cap (B \cup C)$. In either case, it follows that $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$.

($\supset$) Let $x \in (A \cup C) \cap (B \cup C)$ Then $x \in (A \cup C)$ and $x \in (B \cup C)$. So, $x \in A$ or $x \in C$, and $x \in B$ or $x \in C$. If $x \in C$, then $x \in (A \cap B) \cup C$. Otherwise, if $x \notin C$, then $x \in A$ and $x \in B$, so $x \in A \cap B$, so $x \in (A \cap B) \cup C$. In either case, it follows that $(A \cup B) \cap (B \cup C) \subset (A \cap B) \cup C$.

Together, these containments imply that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. □